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PHYSICS OF FLUIDS **25**, 106601 (2013)

Inclined, collisional sediment transport

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We apply the constitutive relations of kinetic theory of granular gases to the transport of cohesionless sediments driven by a gravitational liquid turbulent stream in steady uniform conditions. The sediment-laden flow forms self-equilibrated mechanisms of resistance at the bed surface, below which the sediments are at rest. This geophysical process takes place quite often in streams at moderate slope and may be interpreted through tools common to fluid mechanics and particle physics. Taking into account the viscous dissipation of the fluctuation energy of the particles, and using approximate methods of integration of the governing differential equations, permit to obtain a set of simple formulas for predicting how depths and flow rates adjust to the angle of inclination of the bed, without requiring additional tuning parameters besides the particle and fluid properties. The agreement with laboratory experiments performed with either plastic cylinders or gravel in water is remarkable. We also provide quantitative criteria to determine the range of validity of the theory, i.e., the values of the Shields number and the angle of inclination of the bed for which the particle stresses can be mostly ascribed to collisional exchange of momentum. [http://dx.doi.org/10.1063/1.4823857]

I. INTRODUCTION

Sediment transport in rivers, i.e., the gravitational, non-Brownian motion of particles totally immersed in a turbulent liquid (so that a layer of clear liquid is always present on top of sediments), has been the subject of hundreds of theoretical and experimental studies. However, the studies on sediment transport at small values of the Shields number (the dimensionless shear stress exerted on the particles by the liquid), i.e., when only a portion of the particles at the surface of a static bed is mobilized, outnumber those at larger values of the Shields number, when a sediment layer of at least one up to several particle diameters in thickness is entrained from the bed and move over it.[1] In the former case the moving grains do not interact with each other, and run over the bed in an irregular way (saltating, rolling), affected by near-to-bed turbulence and contact with static grains, alternating periods of motion, and even prolonged rests.[2–5] In the latter case, at least when the weight of the particles is not entirely supported by turbulent suspension,[6] a substantial role in determining velocity and concentration profiles, and in producing the equilibrium conditions at the bed level, is played by the particle stresses; they are mainly induced by collisional momentum transfer,[7] so that constitutive relations from kinetic theories of granular gases apply.[8–10] In the present work, the two above described regimes for sediment transport are hereafter referred to as ''ordinary bedload'' and ''collisional sediment transport,'' respectively. Recently,[11] a criterion for distinguishing the two regimes, in the case of flows over horizontal plane beds driven by pressure gradient, has been proposed. We anticipate that its extension to the case of inclined sediment transport indicates that a significant portion of the available experimental data on laboratory flumes and rivers[12] can actually be classified as collisional sediment transport, in contrast with the common perception. Hence, the necessity of predicting the flow characteristics in that particular regime.

The solution of the collisional sediment transport, even in the simple case of steady and fully developed flow, requires the integration of a system of differential equations composed by momentum and energy balances, and constitutive relations from kinetic theories and turbulence models. While a full numerical integration of the equations is certainly possible,[7,13,14] approximate methods of integration provide analytical expressions that (i) might prove very useful in view of practical applications, (ii) help in clarifying the basic physics that governs the phenomenon, and (iii) promise straight extension to include other regimes.

Capart and Fraccarollo[15] have recently performed detailed measurements on the inclined, steady, and fully developed flows of plastic cylinders and water over erodible beds in a rectangular channel. They made use of the linearity of the concentration and velocity profiles suggested by the experiments to approximately integrate the aforementioned differential equations. To do that, they introduced the additional assumption that the Richardson number, i.e., the measure of turbulence suppression due to density stratification relative to turbulence generation due to the velocity gradient, is roughly constant in collisional sediment transport, following previous works on suspensions.[16,17]

Here, we employ the trapezium rule[11,18] to approximately solve the already cited set of differential equations governing the steady and fully developed, inclined collisional transport of cohesionless particles – extending therefore the analysis of Ref. 11 to deal with the downslope component of gravity. Apart from the inclusion of the possibility of correlated motion among the particles at high concentration,[19] this work differs from that of Ref. 15 because the assumption of constancy of the Richardson number is removed and substituted with the dependence of the collisional coefficient of restitution (the ratio of the magnitude of post to pre-collisional relative velocity between two colliding particles) on the Stokes number, i.e., the ratio of particle inertia to fluid viscous forces.[20,21] This takes into account the viscous dissipation of the fluctuation energy for the particles at high concentration,[14] and shows that the Richardson number actually decreases with the Shields number, in accordance with the experiments, and tends to an asymptotic value when the effects of turbulent suspension become important. The bed slope and the Shields number concur independently in determining the concentration and the particle velocity distribution and must be both considered to define the domain of existence of the investigated flow regime. To complete the physical description of potential sediment-laden flows, we will also offer the criterion to pinpoint the further transition to debris flow, i.e., the gravitational motion of sediments and liquid in which the clear liquid layer at the top of the sediments disappear.[22]

The present theory is also able to predict the existence of a dense layer at the base of the flow, where the turbulence is suppressed and the fluid shear stress is taken to be negligible. As in Ref. 11, we assume that the top of the bed is localized where the ratio of particle pressure to shear stress is at yield and has a characteristic value (corresponding to a characteristic value for the concentration). Across the dense layer, both concentration and ratio of particle pressure to shear stress vary to reach the values that characterize the bed, while the thickness of the dense layer is in most cases negligible with respect to the thickness of the sediment layer. This explains the recent experimental observation of a discontinuity between the concentration at the base of the sediment layer and that within the bed.[15]

The paper is organized as follows: in Sec. II, we introduce the governing equations and solve them by making use of the trapezium rule and the already mentioned fundamental assumptions; in Sec. III, we make comparisons between the predictions of the theory and the experimental results on the sediment transport of plastic cylinders and gravel in water. Some conclusions and suggestions for future works are presented in Sec. IV.

II. THEORY

The sketch of the flow configuration is depicted in Fig. 1. We focus on the flow of identical, inelastic spheres, of diameter d and density ρ_p, over an erodible bed. Particles are driven into collisions by the combined effect of gravity and the shear stress exerted at the top of the particles by a turbulent liquid, of density ρ and molecular viscosity η. We assume that both the particle and the liquid motion are steady and uniform. We limit the analysis to situations in which the height of the particles over the bed, h, is less than the height of the liquid, H. We take x and y to be the direction

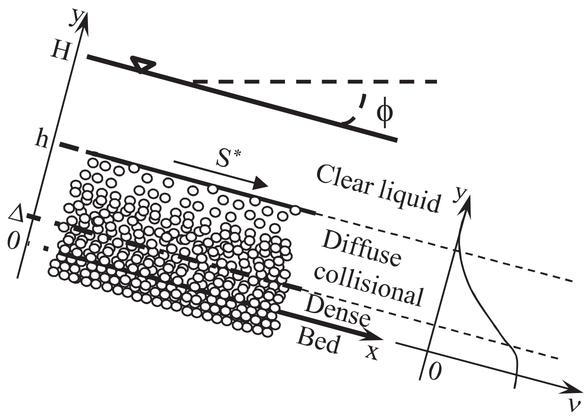

FIG. 1. Sketch of the flow configuration with the frame of reference and a generic concentration profile.

parallel and perpendicular to the bed, respectively, and neglect the variation along the spanwise direction; the bed, located at $y = 0$, is inclined with respect to the horizontal by an angle ϕ. The local particle velocity and concentration are u and ν, respectively. In what follows, all quantities are made dimensionless using the particle density and diameter, and the reduced gravitational acceleration, $g(\sigma - 1)/\sigma$, where g is the gravitational acceleration and σ the ratio of particle to liquid mass density. With this, the inverse of the dimensionless molecular viscosity of the liquid is the particle Reynolds number R.

The balances of particle momentum parallel and perpendicular to the flow are

$$s' = -\frac{\sigma \nu}{\sigma - 1} \sin\phi - D \tag{1}$$

and

$$p' = -\nu \cos\phi, \tag{2}$$

respectively, where s is the particle shear stress, p is the particle pressure, and D is the drag force exerted on the particles by the fluid, which obviously is null where the concentration vanishes. Here, and in what follows, a prime indicates a derivative along y.

The balance of fluid momentum parallel to the flow is simply

$$S' = -\frac{1 - \nu}{\sigma - 1} \sin\phi + D, \tag{3}$$

where S is the fluid shear stress.

We take the particle stresses to vanish at $y = h$. Then, upon substituting $\nu = -p'/\cos\phi$ from Eq. (2) into Eqs. (1) and (3), and integrating, we obtain

$$s + S = S^* + p\tan\phi + \frac{1}{\sigma - 1} \sin\phi\,(h - y)\,, \tag{4}$$

with $S^* = (H - h)\sin\phi/(\sigma - 1)$ the liquid shear stress at the top of the particles. The particle flow is characterized by the presence of an upper layer, in which constitutive relations of classic kinetic theories apply (the diffuse collisional layer), and a lower dense layer, in which the particle velocity fluctuations are correlated and extended kinetic theory applies (Fig. 1).

In the dense layer, as already mentioned, the turbulence of the liquid is suppressed by the presence of the particles, and, therefore, plays no role in determining the particle velocity fluctuations.[11] There, it suffices to use the relation proposed in Ref. 20 for the dependence of the restitution coefficient e on the Stokes number, $\mathrm{St} \equiv \sigma T^{1/2}\mathrm{R}/9$, where T is the granular temperature (one-third the mean square of the particle velocity fluctuations), to capture the influence of the viscous liquid on the particle collisions:

$$e \equiv \varepsilon - 6.9\frac{1 + \varepsilon}{\mathrm{St}}, \tag{5}$$

where ε is the effective coefficient of collisional restitution of the particles in absence of the liquid, which also takes into account the role of friction.[23]

In the following, we will obtain relations between the quantities of interest evaluated at the interface between the dense and the diffuse collisional layer, using the results of kinetic theory (a detailed derivation of the following relations is reported in Refs. 18 and 24). We initiate the theoretical procedure from the value for the coefficient of restitution at $y = \Delta$, e_Δ. When the energy flux can be neglected in the balance of fluctuation energy for the particles, the value, k, of the ratio of particle shear stress to particle pressure at $y = \Delta$ is a function only of the value of e_Δ,[18,24]

$$k = \left(\frac{24J}{5\pi} \frac{1-e_\Delta}{1+e_\Delta} \right)^{1/2}, \tag{6}$$

where, in the dense limit, $J = (1+e_\Delta)/2 + (\pi/4)(3e_\Delta - 1)(1+e_\Delta)^2 / [24 - (1-e_\Delta)(11-e_\Delta)]$. The value of this stress ratio at $y = \Delta$ permits the evaluation of the concentration, ν_Δ, there through

$$\nu_\Delta = \frac{0.6G_\Delta}{0.63 + G_\Delta}, \tag{7}$$

being G_Δ the product of ν_Δ and the radial distribution function, given in Ref. 24 as

$$G_\Delta = \left[\frac{192}{25\pi^{3/2}} \frac{J^2(1-e_\Delta)}{c(1+e_\Delta)^2 k^3} \right]^3, \tag{8}$$

in which c is a material coefficient of order unity.[19] As anticipated, the value of the particle stress ratio at $y = \Delta$ predicted by Eq. (6) is different from the value at which the bed yields (taken to be constant and equal to α in the present work), despite the fact that Δ is small. Such details of the vertical structure of the flow are not described in previous mechanistic views of sediment transport.[25–29] With e_Δ from Eq. (5) and the definition of the Stokes number, we calculate the granular temperature at $y = \Delta$:

$$T_\Delta = \left(\frac{62.1}{\sigma \mathrm{R}} \frac{1+\varepsilon}{\varepsilon - e_\Delta} \right)^2. \tag{9}$$

Then, from the constitutive relation for the pressure of kinetic theories,[9] we evaluate the pressure at $y = \Delta$:

$$p_\Delta = 2\nu_\Delta G_\Delta (1+e_\Delta) T_\Delta. \tag{10}$$

Assuming that the concentration varies linearly from zero at $y = h$ to the value ν_Δ at $y = \Delta$, Eq. (2) gives

$$h - \Delta = \frac{2p_\Delta}{\nu_\Delta \cos\phi}. \tag{11}$$

From Eq. (4), the approximation $p_0 = p_\Delta + \nu_\Delta \Delta \cos\phi$, and the yielding condition $s_0/p_0 = \alpha$, we obtain the depth of the dense layer:

$$\Delta = \frac{p_\Delta}{\nu_\Delta \cos\phi} (k - \alpha) \left[(\alpha - \tan\phi) - \frac{1}{\nu_\Delta(\sigma - 1)} \tan\phi \right]^{-1}, \tag{12}$$

and, with Eq. (11), the total particle depth:

$$h = \frac{p_\Delta}{\nu_\Delta \cos\phi} \left\{ 2 + (k - \alpha) \left[(\alpha - \tan\phi) - \frac{1}{\nu_\Delta(\sigma - 1)} \tan\phi \right]^{-1} \right\}. \tag{13}$$

Using Eq. (11) in Eq. (4), and neglecting S with respect to s in the dense layer,[30] gives

$$S^* = p_\Delta \left[k - \tan\phi - \frac{2}{\nu_\Delta(\sigma - 1)} \tan\phi \right]. \tag{14}$$

The expression for the particle velocity at $y = h$ is that derived in Ref. 11 using the trapezium rule in the constitutive relation for the particle shear stress of kinetic theories,

$$u_h = \frac{5\pi^{1/2}(1+e_\Delta)}{8J_\Delta} k T_\Delta^{1/2} h, \tag{15}$$

here simplified by assuming that the particle velocity vanishes at $y = \Delta$ and neglecting Δ with respect to h. The volume flow rate per unit width of the particles is[11]

$$q = \frac{1}{4}\nu_\Delta u_h h. \tag{16}$$

Also, we can evaluate the liquid depth

$$H = h + \frac{S^*(\sigma - 1)}{\sin\phi}, \tag{17}$$

and, therefore, the Shields number

$$\theta = \frac{H\tan\phi}{\sigma - 1}, \tag{18}$$

here defined without the correction for the bed slope introduced in Ref. 26.

Finally, if we assume that the velocity of the liquid is equal to that of the particles for $y \leq h$, and constant in the upper clear liquid layer, the volume flow rate of the liquid is

$$Q = \frac{2-\nu_\Delta}{\nu_\Delta} q + u_h(H - h). \tag{19}$$

III. DISCUSSION AND COMPARISONS

Here, we make comparisons between the results of the present theory and the experiments performed by Capart and Fraccarollo[15] with plastic cylinders of equivalent diameter equal to 3.35 mm and water (R = 350 and σ = 1.51). As appropriated for plastic beads,[11] we use $\alpha = 0.5$, $\varepsilon = 0.6$, and $c = 0.5$. Actually, only the parameter c is not a directly measurable property. However, its influence on the solution has been discussed in Ref. 11. We emphasize that here we use the value for c inferred from comparisons with a data set of experiments performed on a flow configuration different from that of Ref. 15, without additional tuning.

Figure 2 shows the behavior of the concentration at the bottom of the diffuse collisional layer as a function of the Shields number. The qualitative trend is well reproduced by the theory, although ν_Δ is overestimated at the lowest values of θ and underestimated at the largest. This may be due to the fact that the present definition of Δ (the thickness of the region in which the particle motion is correlated) does not actually coincide with that of Capart and Fraccarollo (the region in which the particle velocity profile is not linear) or to the way ν_Δ is obtained in their work (intercepting at y $= \Delta$ the linear interpolation of the concentration-profile). Figure 2 seems to suggest that ν_Δ can be taken to be approximately constant (and equal to 0.5), independent of the Shields number and the bed slope. Similarly, although not shown here for sake of brevity, the theoretical value of the particle stress ratio at $y = \Delta$ can also be approximated as constant (equal to 0.7). Nonetheless, the capability of the model to detect a bottom boundary condition for the diffuse collisional layer different from that at the bed is a noteworthy feature, which only a detailed experimental research has recently pointed out.[15]

Figure 3 depicts the particle flow rate as a function of the Shields number. The agreement between the present theory and the experiments is remarkable. It is worth emphasizing the minor role played by the bed slope in the determination of the particle flow rate, as suggested by the theoretical curves of Fig. 3 and confirmed by the collapse of the experimental points.

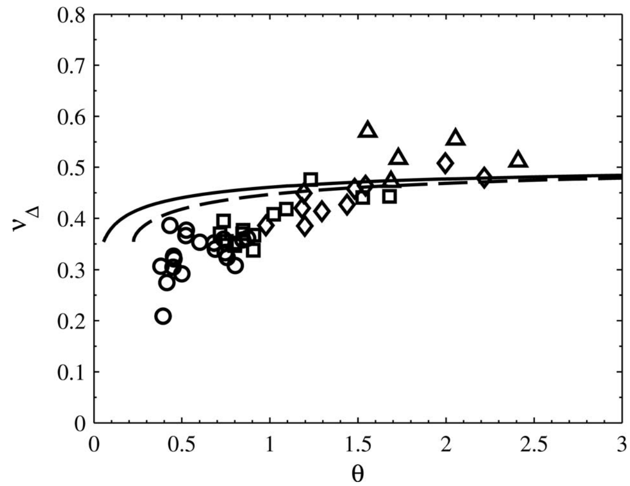

FIG. 2. Experimental (symbols, after Ref. 15) values of concentration at the bottom of the diffuse collisional layer versus Shields number for plastic particles in water when: $0.5^\circ \leq \phi \leq 1.5^\circ$ (circles); $1.5^\circ \leq \phi \leq 2.5^\circ$ (squares); $2.5^\circ \leq \phi \leq 3.5^\circ$ (diamonds); and $3.5^\circ \leq \phi \leq 4.5^\circ$ (triangles). The lines represent the predictions of the present theory when: $\phi = 1.0^\circ$ (solid line); and $\phi = 4.0^\circ$ (dashed line).

From Eqs. (12)–(14) and Eqs. (17) and (18), it is possible to express the total flow depth of the particle,

$$h = \left[1 - \frac{\nu_\Delta (k-\alpha)(\sigma-1)}{2\nu_\Delta(\sigma-1)(k-\tan\phi) - 2\tan\phi}\right]\frac{2}{\nu_\Delta(\alpha-\tan\phi)}\theta, \tag{20}$$

the depth of the bottom dense layer,

$$\Delta = \frac{1}{\alpha-\tan\phi}\frac{(k-\alpha)(\sigma-1)}{\nu_\Delta(\sigma-1)(k-\tan\phi)-\tan\phi}\theta, \tag{21}$$

and the liquid depth,

$$H = \frac{\sigma-1}{\tan\phi}\theta, \tag{22}$$

in terms of the Shields number. With ν_Δ and k approximately constant, Eqs. (20) and (21) indicate that there is a linear relationship between the particle depths and θ at a given bed slope. A linear

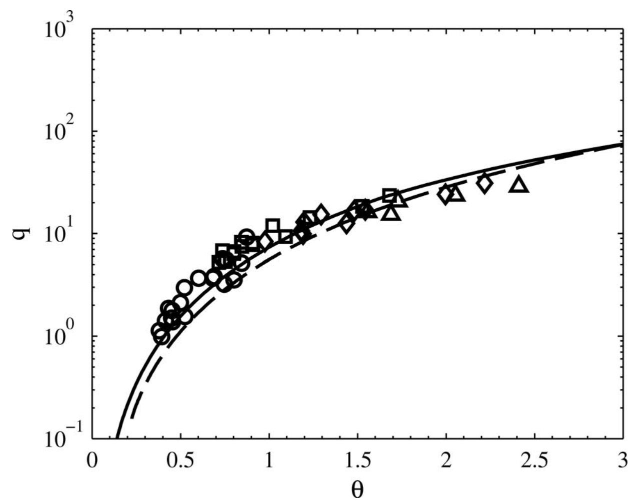

FIG. 3. Same as in Fig. 2, but for the dependence of the particle flow rate on the Shields number.

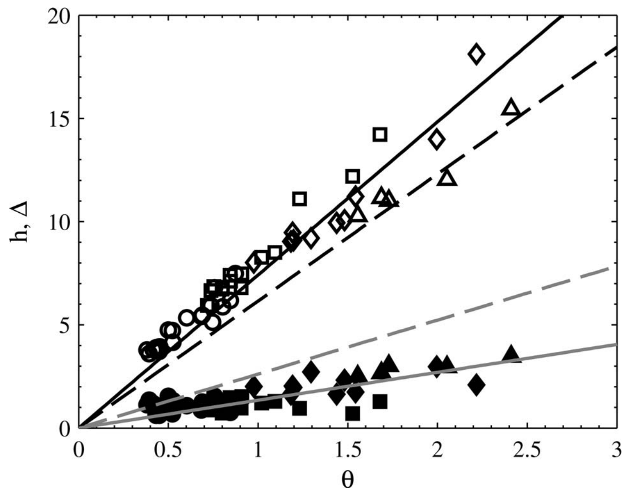

FIG. 4. Same as in Fig. 2, but for the dependence of the particle total flow depth (open symbols) and the depth of the bottom dense layer (filled symbols) on the Shields number. The lines depict the approximate formulas: Eq. (20) (black lines) and Eq. (21) (gray lines), for $\phi = 1.0^\circ$ (solid line) and 4.0° (dashed line).

relation between h and θ has been previously suggested in Ref. 31; however, here we provide an explicit expression for the coefficient of proportionality as a function of the particle properties. Once again, as indicated in Fig. 4, the agreement with the experiments is notable.

The degree of saturation[30,32,33] can be calculated from Eqs. (20) and (22) as

$$\frac{H}{h} = \frac{\nu_\Delta^2 (\sigma - 1)(k - \tan\phi) - \nu_\Delta \tan\phi}{2\nu_\Delta (\sigma - 1)(k - \tan\phi) - 2\tan\phi - \nu_\Delta (k - \alpha)(\sigma - 1)} \frac{(\alpha - \tan\phi)(\sigma - 1)}{\tan\phi}. \tag{23}$$

This formula improves the expression suggested in Ref. 34 for immature ($H \geq h$) debris flows, where a constant coefficient equal to 0.3 was used instead of the first fraction on the right-hand side of Eq. (23), and agrees well with the experiments (Fig. 5(a)).

The ratio of the particle to the liquid flow rate is, from Eqs. (16) and (19),

$$\frac{q}{Q} = \frac{\nu_\Delta}{2 - \nu_\Delta + 4(H/h - 1)}. \tag{24}$$

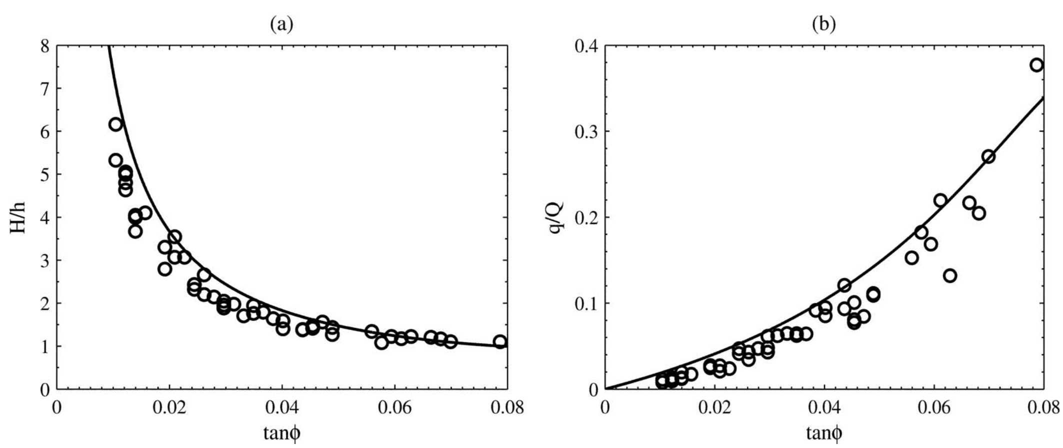

FIG. 5. (a) Experimental (circles, after Ref. 15) and theoretical (solid line, Eq. (23)) degree of saturation as a function of the bed slope for plastic particles in water. (b) Same as in (a), but for the dependence of the ratio of the particle to the liquid flow rate on the bed slope (the solid line represents Eq. (24)).

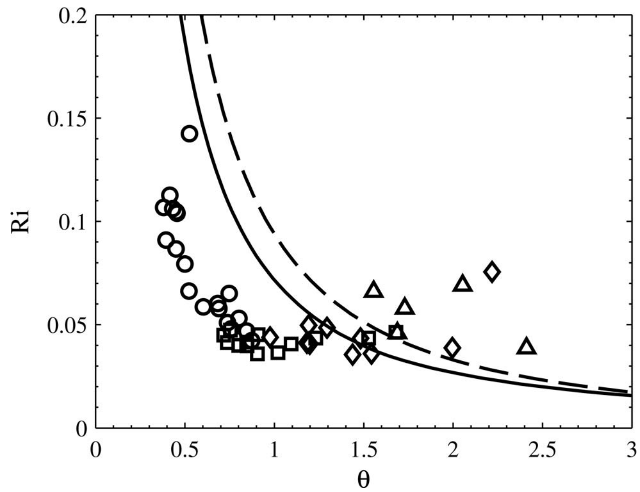

FIG. 6. Same as in Fig. 2, but for the dependence of the Richardson number on the Shields number.

Both the degree of saturation and the flow rate ratio depend only on the bed slope, given the type of particles, with the approximation $\nu_\Delta = 0.5$ and $k = 0.7$. This result is tested against the experiments in Fig. 5(b), in which the experimental data show a remarkable collapse.

As already mentioned, it has been suggested that the Richardson number plays an important role in governing the sediment transport of particles in a turbulent fluid. Figure 6 shows the comparison between the experimental and the theoretical values of the Richardson number, here defined as $\mathrm{Ri} \equiv \sigma \nu_\Delta h \cos\phi / u_h^2$,[15] as a function of the Shields number. The decreasing behavior of the Richardson number with θ, as well as the tendency to saturate to a constant value at large values of the Shields number, when the turbulent suspension is expected to play an important role is well captured by the theory. Both theory and data show a tendency of Ri to diverge at the lowest values of the Shields number, at the interface with the ordinary bedload regime. Although the Richardson number is meaningless for ordinary bedload, its behavior points out the existence of a discontinuity in the physical mechanisms behind the two regimes. The disagreement between theory and experiments reflects the already discussed disagreement on ν_Δ (Fig. 2). However, we emphasize the capability of the model to reproduce the behavior of Ri, a feature which has never been captured before.

We depict hereafter a phase diagram where the domain of existence of the collisional sediment transport is shown. This sediment-laden flow regime comprises gravitational flow of particles entirely immersed in a turbulent liquid for which: (i) the particle stresses are due to collisional momentum transfer and can be modeled using the kinetic theory of granular gases; and (ii) the turbulent suspension of the particles is absent. For given properties of non-cohesive sediments, the present theory allows to characterize the flow regime on the basis of two quantities, the Shields number, θ, and the angle of inclination of the bed, ϕ. At the interface with ordinary bedload, we assume, as in Ref. 11, that the total particle depth must be at least equal to one diameter to fulfill condition (i). With this, we obtain, from Eq. (20) with $h = 1$, the minimum value of the Shields number, given as a function of the bed slope, for which the present theory applies,

$$\breve{\theta} = \left[\frac{\nu_\Delta (\sigma - 1)(k - \tan\phi) - \tan\phi}{2\nu_\Delta (\sigma - 1)(k - \tan\phi) - 2\tan\phi - \nu_\Delta (k - \alpha)(\sigma - 1)} \right] \nu_\Delta (\alpha - \tan\phi) . \quad (25)$$

At large values of the Shields number, the sediment transport ceases to be free from the lifting effects of turbulence. Turbulence begins to suspend the particles at $y = h$ when the liquid shear velocity there, σS^*, is equal to the settling velocity w of a single particle.[6,7] From this and Eqs. (17), (18), (20), and (25), we obtain the value of the Shields number, as a function of the bed slope, for

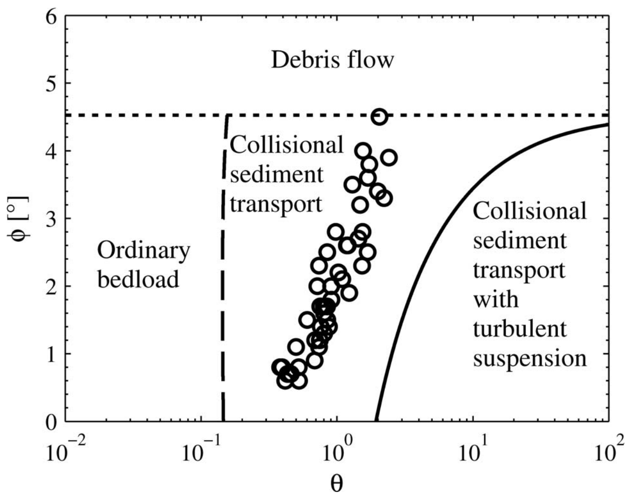

FIG. 7. Regime map for the transport of plastic particles in water. The lines represent the theoretical boundaries (Eq. (25), dashed line; Eq. (26), solid line; Eq. (23) with $H/h = 1$, dotted line), and the circles the experiments of Ref. 15.

the onset of turbulent suspension:

$$\widehat{\theta} = \frac{w^2 \breve{\theta}(\sigma - 1)}{\sigma \left[\breve{\theta}(\sigma - 1) - \tan\phi \right] \cos\phi}. \tag{26}$$

Turbulent suspension can be taken into account in the approximate way suggested in Ref. 6, but its implementation is beyond the scope of the present work.

We can now build a regime map in the θ–ϕ plane to describe the transport of particles in a turbulent liquid (Fig. 7). For the plastic particles employed in Ref. 15, the measured settling velocity was 18 cm/s, corresponding to $w = 1.7$ in dimensionless units. By setting $H/h = 1$ in Eq. (23), we can find the angle of inclination of the bed above which the liquid height is equal or less than the particle depth. This is the realm of debris flows,[30] and we postpone to future works extending the present analysis to deal with that situation. When H is greater than h, and $\theta < \breve{\theta}$, the mean free path between two consecutive collisions is longer than the ballistic trajectory,[11,35] so that it is not possible to disregard the influence of the external forces (gravity, drag, and lift) in describing the micro-mechanical particle-particle interactions: the constitutive relations of kinetic theories of granular gases do not apply. As already mentioned, we call this regime ordinary bedload. The transition between ordinary bedload and collisional sediment transport offers many issues of interests for future works. For example, how do the predictors of sediment transport or some macro-variable, such as the Richardson number, behave across this limit? The present theory does not provide answer to these questions, but helps to face them, at least illuminating one side of the problem.

The region $\breve{\theta} < \theta < \widehat{\theta}$ is the subject of the present theory, i.e., where collisional sediment transport develops. All the experiments of Ref. 15 belong to this region, justifying the notable agreement with the theoretical results shown in the previous plots. When $\theta > \widehat{\theta}$, turbulent suspension plays a role. However, when only a part of the weight of the particles over the bed is supported by the turbulent suspension, a theoretical analysis in the framework of kinetic theories is still possible.[6]

Finally, we make comparisons between the present theory and the experiments reported in Ref. 26 on incline flows of gravel and water with angles of inclination ranging from 1.7° to 11.3°. Smart[26] employed both uniform and non-uniform diameter gravel, with mean diameters ranging from 2 to 10 mm. In the theory, we set $\sigma = 2.65$ and $R = 1150$ (obtained using $d = 6$ mm, the average value). As appropriate for sand or gravel,[6,11] we use $\alpha = 0.5$, $\varepsilon = 0.45$, and $c = 0.5$. As for plastic particles, $\nu_\Delta = 0.5$ and $k = 0.7$ are also good approximations in this case. With these, we build the

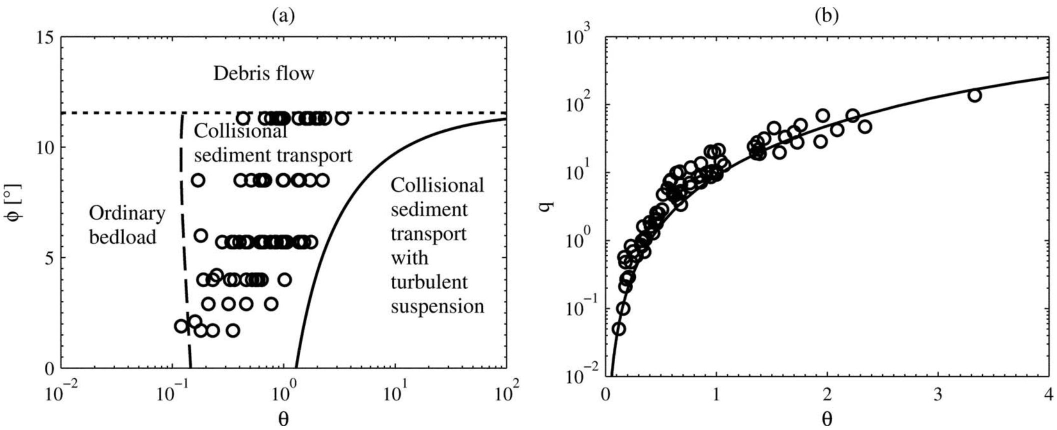

FIG. 8. (a) Same as in Fig. 7, but for the transport of gravel in water (the circles represent the experiments of Ref. 26). (b) Experimental (circles, after Ref. 26) and theoretical (solid line, Eq. (16) with ϕ equal to the experimental average value of 6.5°) particle flow rates as functions of the Shields number for gravel in water.

regime map of Fig. 8(a), using $w = 1.85$,[36] and notice that all, except for one, of the experiments of Ref. 26 can be classified as collisional sediment transport. As expected, the present theory is capable of notably reproducing the experimental data in terms of particle flow rate against the Shields number (Fig. 8(b)).

IV. CONCLUDING REMARKS

In this paper, the steady and fully developed inclined flow of sediments immersed in a turbulent liquid over an erodible bed has been modeled in the frame of kinetic theory of granular gases, accounting for the viscous dissipation of the fluctuation energy of the particles in the high concentrated region near the bed. This differs from the work of Capart and Fraccarollo,[15] where the emphasis was on the competition between turbulence suppression due to the density gradient and turbulence generation due to the velocity gradient. The main results are: (i) the approximate integration of the governing differential equations permits to obtain simple analytical formulas for the dependence of particle and liquid depths and flow rates on the Shields number and the angle of inclination of the erodible bed; (ii) a few, measurable particle and liquid properties are required as inputs to the model, without the need for additional tuning parameters; (iii) the agreement between the theory and the laboratory experiments performed with either plastic cylinders or gravel and water is notable; (iv) the range of Shields numbers for which the present theory applies (collisional sediment transport) increases with the angle of inclination of the erodible bed. This regime is enclosed between ordinary bedload and the transport of sediments fully suspended by turbulence, where constitutive relations of kinetic theory do not hold. The theory is able to provide all the single predictors of interest in sediment transport. Its extension beyond the identified limits, i.e., at larger values of the Shields number, where turbulent suspension plays a role, and at degree of saturation equal or greater than one, in the realm of debris flows, seems feasible and will be the subject of future works.

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